Applied Mathematics
Comprehensive Reference Guide

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Abstract

This document serves as a comprehensive, multi-page reference guide for Applied Mathematics, compiled and expanded from my original linear algebra notes. It is structured to provide clear, categorized, and expanded formulas, theorems, and mathematical derivations across numerous mathematical disciplines.

Contents

Calculus & Real Analysis

Differential Calculus

Derivatives & Theorems

Derivative Rules

Product Rule: $(fg)' = f'g + fg'$ Quotient Rule: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

Chain Rule: $(f \circ g)' = f'(g(x))g'(x)$

Example / Derivation

Chain Rule and Quotient Rule combined: Let $h(x) = \frac{\sin(x^2)}{e^{3x}}$.

$$h'(x) = \frac{[\sin(x^2)]'e^{3x} - \sin(x^2)[e^{3x}]'}{(e^{3x})^2} = \frac{2x\cos(x^2)e^{3x} - 3e^{3x}\sin(x^2)}{e^{6x}} = \frac{2x\cos(x^2) - 3\sin(x^2)}{e^{3x}}$$

Standard Derivatives

- $\frac{d}{dx}x^n = nx^{n-1}$
- $\frac{d}{dx}e^x = e^x$; $\frac{d}{dx}a^x = a^x \ln a$
- $\frac{d}{dx}\ln x = \frac{1}{x}$; $\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$
- $\frac{d}{dx}\sin x = \cos x$; $\frac{d}{dx}\cos x = -\sin x$
- $\frac{d}{dx}\tan x = \sec^2 x$
- $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$
- $\frac{d}{dx}\sinh x = \cosh x$

Taylor / Maclaurin Series

Polynomial approximations of complex functions near a specific point a in the domain.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Common Expansions ($a = 0$):

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Integral Calculus

Integrals & Methods

Fundamental Theorem of Calculus

1st Part: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ 2nd Part (Evaluation): $\int_a^b f(x)\,dx = F(b) - F(a)$

Applications of Integration

Arc Length: $L = \int_a^b \sqrt{1 + (y')^2}\,dx$ or for polar $r(\theta)$: $L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$

Surface Area: $S = 2\pi \int_a^b y\sqrt{1 + (y')^2}\,dx$

Centroids (Center of Mass): $\bar{x} = \frac{1}{A}\int x f(x)\,dx$, $\bar{y} = \frac{1}{2A}\int [f(x)]^2\,dx$

Example / Derivation

Volume of Revolution for $y = x^2$ around x-axis on $[0, 1]$ using the Disk Method:

$$V = \pi \int_0^1 (x^2)^2 \, \mathrm{d}x = \pi \int_0^1 x^4 \, \mathrm{d}x = \pi \left[\frac{x^5}{5}\right]_0^1 = \frac{\pi}{5}$$

Integral Techniques

By parts: $\int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u$

Example / Derivation

Reduction formula for $\int \sin^n x \, \mathrm{d}x$ using integration by parts $(u = \sin^{n-1} x, \ \mathrm{d}v = \sin x \, \mathrm{d}x)$:

$$\int \sin^n x \, \mathrm{d}x = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, \mathrm{d}x$$

Gamma & Beta Functions

Gamma: $\Gamma(n) = \int_0^\infty t^{n-1} \mathrm{e}^{-t} \, \mathrm{d}t$. Properties: $\Gamma(n+1) = n\Gamma(n) = n!$, $\Gamma(1/2) = \sqrt{\pi}$

Beta: $B(m, n) = \int_0^1 t^{m-1}(1-t)^{n-1} \, \mathrm{d}t = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Multivariate & Vector Calculus

Vector Analysis

Operators

Gradient: $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

Divergence: $\operatorname{div} \boldsymbol{F} = \nabla \cdot \boldsymbol{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$

Curl: $\operatorname{curl} \boldsymbol{F} = \nabla \times \boldsymbol{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$

Integration Theorems & Divergence Evaluation

Stokes' Theorem: $\oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \iint_S (\nabla \times \boldsymbol{F}) \cdot \mathrm{d}\boldsymbol{S}$

Divergence Theorem: ${}_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \iiint_V (\nabla \cdot \boldsymbol{F}) \, \mathrm{d}V$

Example / Derivation

Evaluate the line integral $\oint_C (x^2 y \, \mathrm{d}x - xy^2 \, \mathrm{d}y)$ where C is the circle $x^2 + y^2 = R^2$ using Green's Theorem:

$$P = x^2 y, \ Q = -xy^2 \implies \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -y^2 - x^2 = -(x^2 + y^2)$$

$$\oint_C P \, \mathrm{d}x + Q \, \mathrm{d}y = \iint_D -(x^2 + y^2) \, \mathrm{d}A = \int_0^{2\pi} \int_0^R -(r^2) r \, \mathrm{d}r \, \mathrm{d}\theta = -2\pi \left[\frac{r^4}{4}\right]_0^R = -\frac{\pi R^4}{2}$$

Linear Algebra & Differential Equations

Linear Algebra

Matrices & Vector Spaces

Inverses & Orthogonality
Rank-Nullity Theorem: $\text{rank}(A) + \text{nullity}(A) = n$
Adjugate inverse: $A^{-1} = \frac{1}{\det A}\,\text{adj}(A)$
Gram-Schmidt & SVD
Gram-Schmidt Process (Orthogonalization):

$$\boldsymbol{u}_1 = \boldsymbol{v}_1, \quad \boldsymbol{u}_k = \boldsymbol{v}_k - \sum_{j=1}^{k-1} \text{proj}_{\boldsymbol{u}_j}(\boldsymbol{v}_k) = \boldsymbol{v}_k - \sum_{j=1}^{k-1} \frac{\langle \boldsymbol{v}_k, \boldsymbol{u}_j \rangle}{\langle \boldsymbol{u}_j, \boldsymbol{u}_j \rangle} \boldsymbol{u}_j$$

Singular Value Decomposition (SVD): Any matrix $A \in \mathbb{R}^{m \times n}$ can be decomposed as $A = U\Sigma V^\top$, where U and V are orthogonal, and Σ is diagonal with singular values $\sigma_i = \sqrt{\lambda_i(A^\top A)}$.

Example / Derivation

Trace and Determinant invariant under similarity transformation $A = PDP^{-1}$:

$$\text{tr}(A) = \text{tr}(PDP^{-1}) = \text{tr}(P^{-1}PD) = \text{tr}(D) = \sum \lambda_i$$

$$\det(A) = \det(P)\det(D)\det(P^{-1}) = \det(D) = \prod \lambda_i$$

Ordinary Differential Equations

ODE Solutions

First Order ODEs
Linear: $y' + P(x)y = Q(x)$. Multiply by Integrating factor $\mu(x) = \mathrm{e}^{\int P(x)\,\mathrm{d}x}$. Solution: $y = \frac{1}{\mu}\int \mu Q\,\mathrm{d}x + \frac{C}{\mu}$.
Exact: $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$. Exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Example / Derivation

Solve the Bernoulli ODE: $y' + \frac{1}{x}y = xy^2$. Division by y^2 gives $y^{-2}y' + \frac{1}{x}y^{-1} = x$. Sub $v = y^{-1}$, $v' = -y^{-2}y' \implies -v' + \frac{1}{x}v = x \implies v' - \frac{1}{x}v = -x$. Integrating factor $\mu = \mathrm{e}^{\int -1/x\,\mathrm{d}x} = x^{-1}$.

$$\frac{1}{x}v' - \frac{1}{x^2}v = -1 \implies \left(\frac{v}{x}\right)' = -1 \implies \frac{v}{x} = -x + C \implies v = Cx - x^2$$

Finally, $y = \frac{1}{Cx - x^2}$.

Variation of Parameters (2nd Order Linear)
For $y'' + p(x)y' + q(x)y = g(x)$, with homogeneous solution $y_h = c_1y_1 + c_2y_2$:

$$y_p(x) = -y_1 \int \frac{y_2 g(x)}{W}\,\mathrm{d}x + y_2 \int \frac{y_1 g(x)}{W}\,\mathrm{d}x$$

where Wronskian $W(y_1, y_2) = y_1y_2' - y_2y_1'$.

Partial Differential Equations

PDE Solutions

Wave Equation ($u_{tt} = c^2 u_{xx}$)
d'Alembert's Solution (Initial conditions: $u(x,0) = f(x), u_t(x,0) = g(x)$):

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s$$

Example / Derivation

Heat Equation $u_t = \alpha^2 u_{xx}$ with Separation of Variables $u(x,t) = X(x)T(t)$:

$$X(x)T'(t) = \alpha^2 X''(x)T(t) \implies \frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2$$

Yields PDEs: $T'(t) + \alpha^2\lambda^2 T(t) = 0 \implies T(t) = C\mathrm{e}^{-\alpha^2\lambda^2 t}$
And $X''(x) + \lambda^2 X(x) = 0 \implies X(x) = A\cos(\lambda x) + B\sin(\lambda x)$.
Superposition principle builds the full Fourier series solution.

Transforms & Complex Analysis

Laplace & Fourier Transforms

Integral Transforms

Laplace Transform & System Analysis

Transform: $\mathcal{L}\{f(t)\}(s) = F(s) = \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t$

Convolution Theorem: $\mathcal{L}\{(f*g)(t)\} = F(s)G(s)$ where $(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau$

Example / Derivation

Using Laplace for Damped Harmonic Oscillator: $x'' + 2\zeta\omega_n x' + \omega_n^2 x = 0$ with $x(0) = x_0, x'(0) = v_0$.

$$[s^2X(s) - sx_0 - v_0] + 2\zeta\omega_n[sX(s) - x_0] + \omega_n^2 X(s) = 0$$

$$X(s)(s^2 + 2\zeta\omega_n s + \omega_n^2) = (s + 2\zeta\omega_n)x_0 + v_0 \implies X(s) = \frac{(s+2\zeta\omega_n)x_0 + v_0}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

Inverse transforming gives the physical damped sine-wave trajectory.

Fourier Series & Transform

Fourier Series: $f(x) = \frac{a_0}{2} + \sum_{n=1}^\infty \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$

Coefficients: $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x$

Fourier Transform: $\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t \implies f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega$

Complex Analysis

Functions of a Complex Variable

Cauchy-Riemann & Analytic Functions

$f(z) = u(x,y) + \mathrm{i}v(x,y)$ is analytic if:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \implies \nabla^2 u = 0 \text{ and } \nabla^2 v = 0 \text{ (Harmonic)}$$

Cauchy's Theorems & Residues

Integral Theorem: $\oint_C f(z)\,\mathrm{d}z = 0$ for analytic f within C.

Integral Formula: $f^{(n)}(a) = \frac{n!}{2\pi\mathrm{i}}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,\mathrm{d}z$

Residue Theorem: $\oint_C f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\sum \mathrm{Res}(f, z_k)$

Example / Derivation

Evaluate $I = \int_{-\infty}^{\infty}\frac{1}{x^2+1}\,\mathrm{d}x$ using Contour Integration.
Consider the contour C consisting of the real axis from $-R$ to R and a semicircle Γ_R in the upper half plane. Let $f(z) = \frac{1}{z^2+1} = \frac{1}{(z-\mathrm{i})(z+\mathrm{i})}$.
Pole within contour is at $z_0 = \mathrm{i}$. The residue is:

$$\mathrm{Res}(f, \mathrm{i}) = \lim_{z\to\mathrm{i}}(z-\mathrm{i})\frac{1}{(z-\mathrm{i})(z+\mathrm{i})} = \frac{1}{2\mathrm{i}}$$

By Residue theorem: $\oint_C f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\left(\frac{1}{2\mathrm{i}}\right) = \pi$. Since $\int_{\Gamma_R} f(z)\,\mathrm{d}z \to 0$ as $R\to\infty$, $I = \pi$.

Probability, Statistics & Mechanics

Probability & Statistics

Stochastics

Bayesian Inference and Expectation

Bayes: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

Expected Value (Mean): $E[X] = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x$ $\quad \mathrm{Var}(X) = E[X^2] - (E[X])^2$

Distributions & Regression

Normal $N(\mu, \sigma^2)$: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Poisson Poi(λ): $P(X = k) = \frac{\lambda^k \mathrm{e}^{-\lambda}}{k!}$

Simple Linear Regression ($y = \beta_0 + \beta_1 x + \varepsilon$):

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)}, \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

Example / Derivation

Moment Generating Function (MGF) for Poisson Distribution ($X \sim \mathrm{Poi}(\lambda)$):

$$M_X(t) = E[\mathrm{e}^{tX}] = \sum_{k=0}^{\infty} \mathrm{e}^{tk} \frac{\lambda^k \mathrm{e}^{-\lambda}}{k!} = \mathrm{e}^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda \mathrm{e}^t)^k}{k!}$$

Using Maclaurin series for e^x, the sum equates to $\mathrm{e}^{\lambda \mathrm{e}^t}$. Thus, $M_X(t) = \mathrm{e}^{\lambda(\mathrm{e}^t - 1)}$. Taking the first derivative $M_X'(t = 0)$ yields the mean: $E[X] = \lambda$.

Classical Mechanics & Fluid Dynamics

Physical Equations

Newton & Classical Mechanics

Newton's Second Law ($\boldsymbol{F} = m\ddot{\boldsymbol{r}}$) and Central Force (Gravity/Electrostatics):

$$F(r) = -\frac{k}{r^2} \implies m\left(\ddot{r} - r\dot{\theta}^2\right) = F(r), \quad L = mr^2\dot{\theta} = \text{const}$$

Yields the specific orbit equation for $u(\theta) = 1/r(\theta)$:

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2u^2} F(1/u)$$

Fluid Dynamics: Navier-Stokes & Bernoulli

Navier-Stokes (Viscous, incompressible flow velocity $\boldsymbol{u}$):

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\right) = -\nabla p + \mu \nabla^2 \boldsymbol{u} + \rho \boldsymbol{g}$$

If steady ($\partial/\partial t = 0$), inviscid ($\mu = 0$), and incompressible (ρ=const) along a streamline:

$$p + \frac{1}{2}\rho|\boldsymbol{u}|^2 + \rho g z = \text{constant} \quad \text{(Bernoulli's Equation)}$$

Example / Derivation

Continuity Equation (Conservation of Mass): $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0$. For incompressible liquids, density ρ is constant, yielding:

$$\nabla \cdot \boldsymbol{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0 \quad \text{(Divergence-Free Vector Field)}$$

Advanced Topics & Computations

Numerical Methods Finance

Computational Approximations & Stock Modeling

Runge-Kutta 4th Order (RK4) for ODE $y' = f(x, y)$

$$k_1 = hf(x_n, y_n), \quad k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right), \quad k_4 = hf(x_n + h, y_n + k_3)$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Financial Mathematics: Black-Scholes

Models the price variation of underlying financial instruments over time with Geometric Brownian Motion (GBM):

$$\mathrm{d}S_t = \mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t$$

The Black-Scholes PDE for a derivative price $V(S, t)$:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Example / Derivation

Applying Itô's Lemma to find the analytical solution to GBM. Let $f(S, t) = \ln S$. According to Itô calculus for $f(S, t) = \ln S_t$:

$$\mathrm{d}(\ln S_t) = \frac{1}{S_t}\,\mathrm{d}S_t - \frac{1}{2S_t^2}(\mathrm{d}S_t)^2 = \frac{1}{S_t}(\mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t) - \frac{1}{2S_t^2}(\sigma^2 S_t^2 \,\mathrm{d}t)$$

$$\mathrm{d}(\ln S_t) = \left(\mu - \frac{\sigma^2}{2}\right)\mathrm{d}t + \sigma \,\mathrm{d}W_t$$

Integrating from 0 to t gives the exact stock trajectory $S_t = S_0 \exp\big((\mu - \sigma^2/2)t + \sigma W_t\big)$.

Number Theory & Inequalities

Discrete Mathematics

Modular Arithmetic & RSA

Euler's Totient $\phi(n)$: The number of integers up to n coprime to n. Euler's Theorem states $a^{\phi(n)} \equiv 1 \pmod{n}$.

RSA Encryption: Choose public key e, private key d such that $ed \equiv 1 \pmod{\phi(n)}$.

$$\text{Ciphertext } C \equiv M^e \pmod{n} \quad \text{Plaintext } M \equiv C^d \pmod{n}$$

Core Inequalities

AM-GM: $\frac{1}{n}\sum_{i=1}^n x_i \geq \left(\prod_{i=1}^n x_i\right)^{1/n}$ Cauchy-Schwarz: $\left(\sum_{i=1}^n a_i b_i\right)^2 \leq \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right)$

Example / Derivation

Proving the Triangle Inequality using Cauchy-Schwarz inequality:

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2 = \langle \boldsymbol{u}+\boldsymbol{v}, \boldsymbol{u}+\boldsymbol{v}\rangle = \|\boldsymbol{u}\|^2 + 2\langle \boldsymbol{u}, \boldsymbol{v}\rangle + \|\boldsymbol{v}\|^2$$

By Cauchy-Schwarz, $\langle \boldsymbol{u}, \boldsymbol{v}\rangle \leq |\langle \boldsymbol{u}, \boldsymbol{v}\rangle| \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\|$. Substituting this:

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2 \leq \|\boldsymbol{u}\|^2 + 2\|\boldsymbol{u}\|\|\boldsymbol{v}\| + \|\boldsymbol{v}\|^2 = (\|\boldsymbol{u}\| + \|\boldsymbol{v}\|)^2$$

Taking the square root yields the Triangle Inequality: $\|\boldsymbol{u}+\boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$.