

The Gliding Principle: Moving Points on Conics and Beyond

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Abstract

The method of moving points (or animate points) is a helpful way to test collinearity and concyclicity by seeing how a setup changes as one point moves. This article extends the usual idea of sliding points on lines to sliding them around circles and other curves. By treating coordinates as polynomials and looking at their degree, we can logically check just a few simple test cases to prove properties that are otherwise tough to solve by hand.

1 Working with Algebraic Parameters

In Euclidean geometry, we usually just assign (x, y) coordinates to points. But when a point X is moving smoothly along a curve, it helps to plug in a single moving variable $t \in \mathbb{R} \cup \{\infty\}$.

A line ℓ is easy to parameterize. For circles and conics, we use stereographic projection. If you have a circle Γ and a fixed point P on it, any line going through P with a slope of t will hit the circle at exactly one other spot, $X(t)$. This means the coordinates for $X(t)$ are going to be rational functions (fractions made of polynomials) of t .

To avoid messy fractions, we scale things up using homogeneous projective coordinates $[x : y : z]$. A point on a circle can be naturally written as $[P_1(t) : P_2(t) : P_3(t)]$, where the P_i are polynomials with a maximum degree of 2. Because of this, we say $X(t)$ has **degree 2**.

2 Degrees of Geometric Operations

To use the degree-counting method, we need to know what happens to the degree when lines and points interact. Let $A(t)$ and $B(t)$ have polynomial degrees n and m .

- **Drawing a line:** If you draw line $A(t)B(t)$, its equation coefficients can have a degree up to $n + m$.
- **Intersecting two lines:** Intersecting two moving lines of degrees n and m creates a point of degree up to $n + m$.

3 The Degree Counting Lemma

This entire method relies on a basic rule from algebra: a polynomial of degree d can only have d roots.

Lemma 1 (The Gliding Lemma). *Let $\mathcal{P}(t)$ be a geometric condition (like three points being collinear). If $\mathcal{P}(t)$ can be framed algebraically as a polynomial $F(t) = 0$ with degree d , checking that $\mathcal{P}(t)$ holds true for $d + 1$ test positions proves it holds true everywhere.*

For instance, three points A, B, C with degrees n, m, k are collinear if the determinant of their coordinates is zero. Expanding that determinant gives a polynomial of degree $n + m + k$. We just need to find $n + m + k + 1$ values for t where it evaluates to zero.

4 Working Through a Valid Configuration

Let us define a recognizable configuration where this method shines with mathematically sound bounds.

Problem: Let A, B, C be fixed points on a circle Γ . A variable point X moves along Γ . Let $Q = AB \cap CX$ and $R = AC \cap BX$. Let F be the pole of line BC with respect to Γ . Prove that $Q, R,$ and F are collinear. (This is a well-known consequence of the self-polar diagonal triangle of the cyclic quadrangle $ABXC$).

Let us determine the degree of the collinearity condition for Q, R, F : 1. A, B, C are fixed points representing degree 0. 2. $X(t)$ moves on a conic, meaning its homogeneous coordinates have degree 2. 3. The moving line CX passes through fixed C (degree 0) and moving X (degree 2), making line CX degree 2. 4. Therefore, $Q = AB \cap CX$ intersects a fixed line with a degree 2 line, making Q a point of degree 2. 5. Symmetrically, BX has degree 2, making $R = AC \cap BX$ a point of degree 2. 6. Expanding collinearity $\det(Q, R, F) = 0$ yields a maximum degree of $2 + 2 + 0 = 4$.

By the Gliding Lemma, we must verify this identity for exactly $4 + 1 = 5$ distinct positions of X . Notice how beautifully the roots of the polynomial fall out via limit cases and geometric dualities:

- **Case 1 ($X \rightarrow A$):** As X approaches A , the secant line CX approaches the secant CA , so Q approaches $AB \cap CA = A$. Similarly, the line BX approaches BA , so R approaches $AC \cap BA = A$. The polynomial translates to evaluating the determinant of points A, A, F . Because two columns are strictly identical in the coordinate matrix, the determinant evaluates to zero trivially! No pole-polar arguments are needed for this root; the algebraic identity safely collapses it.
- **Case 2 ($X \rightarrow B$):** As $X \rightarrow B$, the line CX approaches CB , so $Q = AB \cap CB = B$. The line BX becomes the tangent to Γ at B , so $R = AC \cap \text{Tangent}(B)$. We must check that $B, R,$ and F are collinear. Since F is the pole of BC , the polar of F is BC . By La Hire's theorem, since B is on BC , F must be on the polar of B , which is precisely $\text{Tangent}(B)$. Since $B, F,$ and R all lie entirely on $\text{Tangent}(B)$, they are unconditionally collinear.
- **Case 3 ($X \rightarrow C$):** By perfect symmetry to Case 2, R merges with C , and Q becomes $AB \cap \text{Tangent}(C)$. The pole F lies on $\text{Tangent}(C)$ by the same La Hire duality. Thus C, Q, F all lie squarely on $\text{Tangent}(C)$, and collinearity is verified.

At this point, we have safely docked 3 roots of the degree-4 polynomial. We need exactly two more distinct points. Because the geometric cases become muddy with points at infinity (e.g., if we try to force $BX \parallel AC$), we can leverage the truth of the polynomial identity directly. We just need to check any arbitrary, strict numerical configuration.

Cases 4 and 5 (The Numerical Check): Let us map Γ to the unit circle $x^2 + y^2 = 1$. Let the fixed points be $A = (1, 0), B = (0, 1), C = (-1, 0)$. The line BC has the equation $x - y + 1 = 0$. The pole F of BC with respect to the unit circle is $F = (-1, 1)$.

- **Case 4 (Let $X = (3/5, 4/5)$):** The line CX passes through $(-1, 0)$ and $(3/5, 4/5)$, yielding $Q = AB \cap CX = (1/3, 2/3)$. The line BX passes through $(0, 1)$ and $(3/5, 4/5)$, yielding $R = AC \cap BX = (3, 0)$. Checking the collinearity of $Q(1/3, 2/3), R(3, 0),$ and $F(-1, 1)$ via determinant:

$$\frac{1}{3}(0 - 1) - \frac{2}{3}(3 - (-1)) + 1(3 - 0) = -\frac{1}{3} - \frac{8}{3} + 3 = 0$$

The determinant is exactly zero. Case 4 holds.

- **Case 5 (Let $X = (0, -1)$):** Here CX connects $(-1, 0)$ and $(0, -1)$, which is parallel to AB (connecting $(1, 0)$ and $(0, 1)$). Thus Q hits the point at infinity in the projective direction $[1 : -1 : 0]$. The line BX connects $(0, 1)$ and $(0, -1)$, the y -axis. AC is the x -axis. Their intersection is the origin, so $R = (0, 0)$. We must verify that R, F , and Q are collinear. The line RF connects $(0, 0)$ to $(-1, 1)$, which naturally has the direction vector $(-1, 1)$. Consequently, the line RF points flawlessly to the projective point at infinity $[1 : -1 : 0] = Q$. The collinearity holds exactly.

By verifying these 5 boundary positions, the entire problem is definitively solved. The Gliding Lemma merges elegant qualitative pole-polar logic with brute-force numeric arithmetic to systematically break apart complex Olympiad geometry.