Proposed Open Problems

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Abstract

The following problems are proposed as open problems for Mathematical Reflections. They cover various areas of problem-solving mathematics including Number Theory, Functional Analysis, Combinatorics, and Geometry. No solutions are provided as they are intended as open challenges.

Problem 1 (Number Theory)**.** *Let p be an odd prime. A sequence of positive integers $(a_n)_{n\geq 1}$ is defined by $a_1 = 2$, and for $n \geq 1$,*

$$a_{n+1} = a_n^p + p^{a_n} \pmod{p^{n+2}}$$

where we take the least positive residue. Determine if there exist infinitely many primes p for which the sequence a_n is eventually strictly increasing.

Problem 2 (Functional Analysis / Equations)**.** *Find all continuous functions $f : (0, \infty) \to (0, \infty)$ such that for all $x, y > 0$, the following relation holds:*

$$f\left(x \int_0^y f(t)dt\right) + f\left(y \int_0^x f(t)dt\right) = f(x+y) + f(xy)$$

Problem 3 (Combinatorics)**.** *Let $n \geq 3$ be an integer. Consider a regular n-gon inscribed in a circle. We draw all all its diagonals, assuming that no three diagonals intersect at a single interior point. The regions formed inside the n-gon are to be colored using k colors such that any two regions sharing a line segment as a boundary have different colors.*

Let $C(n)$ be the minimum number of colors k required. Furthermore, a coloring is called balanced if the difference in the number of regions of any two colors is at most 1. Does there exist an integer N such that for all $n > N$, a balanced valid coloring always requires at least $C(n) + 1$ colors?

Problem 4 (Geometry)**.** *Let $\triangle ABC$ be an acute, scalene triangle with circumcircle Γ and incircle ω. Let I be the incenter. The A-excircle ω_A touches the side BC at D. The circumcircle of $\triangle BIC$ intersects Γ again at P (with $P \neq A$). Let X be the intersection of the tangents to Γ at B and C. The line PI intersects ω at two points, U and V, with U closer to P.*

Prove that if the points A, U, D, X are concyclic, then the center of the circle passing through them must lie on the Euler line of $\triangle ABC$.

Problem 5 (Algebra / Polynomials)**.** *Let $P(x)$ be a polynomial with integer coefficients of degree $d \geq 2$. Suppose that for some integer $m \geq 2$, the polynomial $Q(x) = \underbrace{P(P(\cdots P(x)\cdots))}_{m \text{ times}} - x$ has exactly k distinct integer roots.*

Is it true that for any fixed m, there is an absolute constant C_m such that $k \leq C_m d$? If not, is it possible to construct a polynomial sequence where $\lim_{d\to\infty} \frac{k}{d} = \infty$? Evaluate the supremum of k across all possible polynomials of degree d and give an explicit construction.

Problem 6 (Combinatorial Number Theory)**.** *Let $f(n)$ be the smallest positive integer such that any set of $f(n)$ distinct positive integers contains a subset B of size exactly n satisfying the following two conditions simultaneously:*

1. *The sum of the elements in B is divisible by n.*
2. *The product of the elements in B is a perfect n-th power.*

Does $f(n)$ exist for all integers $n \geq 2$? If so, determine the asymptotic behavior and an upper bound for $f(n)$.