Additional Proposed Open Problems

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Abstract

The following 10 problems are proposed as open problems for Mathematical Reflections. They cover various areas of problem-solving mathematics including Number Theory, Functional Analysis, Combinatorics, Geometry, and Algebra. No solutions are provided as they are intended as open challenges for readers and researchers.

Problem 1 (Number Theory / Arithmetic Dynamics)**.** *Let c be a positive integer. Define the sequence $(x_n)_{n\geq 1}$ by $x_1 = 2$ and $x_{n+1} = x_n^2 - c$ for all $n \geq 1$. Let $\omega(x_n)$ denote the number of distinct prime factors of x_n. Does there exist an absolute constant K (independent of c) such that if c is not of the form $k^2 - 2$, the sequence contains infinitely many terms with $\omega(x_n) \leq K$? What is the minimum possible value of K?*

Problem 2 (Combinatorial Geometry)**.** *Let S be a set of n points in the plane in general position (no three points on a line, no four points on a circle). A triangle formed by three points of S is called empty if its interior contains no other points of S. Let $E_A(S)$ denote the number of empty acute triangles. Is it true that for any sufficiently large n, there exists a configuration S such that $E_A(S) < cn$ for some absolute constant c? Provide an explicit construction or prove a super-linear lower bound for $E_A(S)$.*

Problem 3 (Functional Equations / Real Analysis)**.** *Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,*

$$f(f(x) + y) + f(f(y) - x) = f(x + y) + f(x - y) + 2xf(x)$$

Is it possible to find a strictly non-differentiable (everywhere continuous, nowhere differentiable) function that satisfies a similar functional equation variant?

Problem 4 (Combinatorics / Graph Theory)**.** *Let $G = (V, E)$ be a finite simple graph. An edge coloring of G with k colors is called locally rainbow if every vertex $v \in V$ is incident to at least $\min(\deg(v), k)$ different colors. Let $LR(G)$ be the minimum number of colors required for a locally rainbow coloring of G such that no cycle of length 4 is perfectly bicolored (i.e., its edges alternate between exactly two colors). Prove or disprove that there exists a constant C such that $LR(G) \leq C\sqrt{|V|}$ for any graph G with average degree $O(1)$.*

Problem 5 (Algebra / Polynomial Diophantine). *Do there exist non-constant, coprime polynomials $P(x)$ and $Q(x)$ with integer coefficients such that the polynomial*

$$R(x) = P(x)^{2027} + Q(x)^{2027}$$

has the property that all of its roots are strictly real, and it factors as the product of exactly d distinct irreducible polynomials over $\mathbb{Q}$, where $d \geq 2027$?

Problem 6 (Geometry / Triangle Centers). *Let $\triangle ABC$ be an acute triangle with circumcircle Γ and orthocenter H. Let P be a variable point on Γ. Let the reflections of P across the sides BC, CA, AB be P_a, P_b, P_c respectively. It is well known that P_a, P_b, P_c lie on a line passing through H (the Steiner line). Now, instead of reflections, let P'_a be the point such that $\triangle PBP'_a$ is directly similar to $\triangle ABC$, and cyclically for P'_b and P'_c. Do these three points still exhibit collinearity passing through a fixed classical triangle center as P varies on Γ? If so, determine this center.*

Problem 7 (Combinatorial Number Theory). *Let $A \subset \mathbb{N}$ be a set of positive integers having positive upper density. Is it always possible to find three distinct elements $a, b, c \in A$ such that a divides $b + c$ and b divides $c + a$? What if the condition is tightened such that a, b, c must form an arithmetic progression?*

Problem 8 (Algebra / Inequalities). *Let $n \geq 3$ be an integer. Determine the best possible constant $c_n \in \mathbb{R}$ such that for all positive real numbers $x_1, x_2, \ldots, x_n$ satisfying $\prod_{i=1}^{n} x_i = 1$, the following inequality holds:*

$$\sum_{i=1}^{n} \frac{1}{x_i^2 + x_{i+1} + c_n} \geq \frac{n}{2 + c_n}$$

(where $x_{n+1} = x_1$). Does this constant monotonically approach zero as $n \to \infty$?

Problem 9 (Graph Theory / Game Theory). *Consider a Maker-Breaker game played on the edge set of the complete graph K_n. Maker and Breaker alternately claim 1 and b previously unclaimed edges, respectively. Maker's goal is to build a spanning tree with maximum degree at most k. What is the threshold bias $b^*(n, k)$ for Breaker to win? For fixed k, does $b^*(n, k)$ grow logarithmically or algebraically with n?*

Problem 10 (Number Theory / Sequences). *Let $p > 3$ be a prime. Let $S_1 = \{1, 2, \ldots, p-1\}$. For $n \geq 1$, define S_{n+1} to be the set of all distinct least positive residues modulo p of the products ab, where $a, b \in S_n$ and $a \neq b$. If we define $N_p(n) = |S_n|$, does there exist a uniform upper bound on the number of steps k until S_k reaches a fixed point (i.e., $S_{k+1} = S_k$), and does this fixed point inevitably depend solely on the quadratic non-residues of p?*