Physics Handout 4

Advanced Mechanics & Oscillatory Systems

Theoretical Analysis of Friction, Geometry, and Parametric Driving

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Abstract

This document synthesizes the theoretical concepts underlying a set of advanced mechanics problems (Problems 01–06). The topics covered include friction-driven harmonic motion, geometric non-linearities in restoring forces, the tautochrone problem (Huygens pendulum), Coulomb damping in coupled systems, and the adiabatic invariants of parametric oscillators (variable length pendulums).

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1 Friction-Induced Oscillations (Problem 01)

1.1 Mechanism of Restoring Force

In standard Simple Harmonic Motion (SHM), the restoring force is typically elastic ($F = -kx$). However, in Problem 01 (rod on rotating rollers), the restoring force arises from the **position-dependent normal force** modulating the kinetic friction.

Consider a rod of mass M on two rollers separated by distance a, rotating in opposite directions. When the rod is displaced by x from the center:

- **Torque Balance:** To prevent rotation, the normal forces N_1 and N_2 must adjust. Taking torques about the center of mass (CM):

$$N_1 \left(\frac{a}{2} + x\right) = N_2 \left(\frac{a}{2} - x\right)$$

  Combined with $N_1 + N_2 = Mg$, this yields:

$$N_1 = \frac{Mg}{a} \left(\frac{a}{2} - x\right), \quad N_2 = \frac{Mg}{a} \left(\frac{a}{2} + x\right)$$

- **Friction Differential:** The net force is the difference in kinetic friction $f_k = \mu N$.

$$F_{net} = \mu N_1 - \mu N_2 = -\frac{2\mu Mg}{a} x$$

1.2 Conclusion

Because $F_{net} \propto -x$, the system undergoes SHM despite friction usually being a dissipative force. This is a classic example of **feedback** where the system's geometry converts a constant energy source (rotating rollers) into a conservative-like restoring force.

2 Geometric Linearization & Spring Constants (Problem 02)

2.1 Longitudinal vs. Transverse Stiffness

This problem highlights how the effective spring constant k_{eff} depends on the geometry of displacement.

1. **Longitudinal (Standard):** Displacement along the spring axis directly stretches the bonds.

$$F_{long} = -kx \implies \omega_{long} = \sqrt{k/m}$$

2. **Transverse (Geometric):** Displacement y perpendicular to the spring axis creates a restoring force via tension T.

$$F_{trans} = -2T \sin\theta \approx -2T \frac{y}{L}$$

   Here, tension $T = k(L - L_0)$. If the spring is pre-stretched significantly ($L \gg L_0$), T is roughly constant for small y, and the motion is linear.

$$k_{trans} = \frac{2T}{L} = \frac{2k(L - L_0)}{L}$$

Key Insight

Transverse stiffness is governed by *tension*, whereas longitudinal stiffness is governed by *elasticity*. This is analogous to the difference between sound waves in a solid (elasticity) and waves on a guitar string (tension).

3 Isochronism and the Cycloid (Problem 03)

3.1 The Period of a Simple Pendulum

A simple pendulum is not strictly isochronous. The period T depends on amplitude θ_0:

$$T(\theta_0) \approx 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{16}\theta_0^2 + \dots\right)$$

This non-linearity arises because the restoring force $F \propto \sin\theta$ is "softer" than the required linear $F \propto \theta$ at large angles.

3.2 The Huygens (Cycloidal) Pendulum

To achieve perfect isochronism (T independent of amplitude), the particle must travel along a **tautochrone curve**.

- **Geometry:** The tautochrone is a cycloid.
- **Evolute of a Cycloid:** Huygens utilized the property that the evolute of a cycloid is another cycloid. By wrapping the pendulum string around cycloidal "cheeks," the bob traces a cycloid.
- **Physics:** On a cycloid, the effective arc length s relates to the restoring force such that $F \propto -s$ exactly, not approximately. This ensures ω is constant for any amplitude.

4 Coupled Oscillators & Coulomb Damping (Problem 04)

4.1 Static vs. Kinetic Friction Interface

This problem introduces a **hybrid dynamical system** that switches regimes based on acceleration.

- **Sticking Regime ($a < a_{crit}$):** The blocks move as a rigid body. The friction f_s acts as a constraint force, transmitting acceleration.
- **Slipping Regime ($a > a_{crit}$):** The blocks decouple. Friction becomes a constant external force $f_k = \mu N$ opposing motion relative to the slab.

4.2 Energy Decay via Dry Friction

Unlike viscous damping ($F \propto -v$), which decays amplitude exponentially ($A \propto e^{-\gamma t}$), dry friction (Coulomb damping) removes a fixed amount of energy per cycle:

$$\Delta E_{cycle} = -4 f_k A$$

Since $E \propto A^2$, we have $\Delta(A^2) \propto A\Delta A$. This leads to a **linear decay** of amplitude:

$$A(t) = A_0 - \alpha t$$

The oscillation stops completely in finite time, unlike viscous damping which asymptotically approaches zero.

5 Parametric Resonance & Adiabatic Invariants (Problem 05)

5.1 Variable Parameters in Oscillators

Consider an oscillator where the natural frequency $\omega(t)$ changes slowly with time (e.g., changing pendulum length $l(t)$). The equation of motion becomes:

$$\ddot{\theta} + \omega^2(t)\theta = 0$$

If $\omega(t)$ varies slowly (adiabatic limit, $\dot{\omega}/\omega \ll \omega$), the exact solution is difficult, but the energy evolves predictably.

5.2 Adiabatic Invariant

For a harmonic oscillator with slowly varying parameter λ:

$$I = \oint p\, dq = \frac{E}{\omega} = \text{constant}$$

This is the **Adiabatic Invariant**.

- For a pendulum, $\omega = \sqrt{g/l}$.
- If length l decreases (shortens), ω increases.
- To keep E/ω constant, Energy E must increase.

Physical Interpretation: Pulling the string up requires work against the centripetal component of tension. This work pumps energy into the system.

$$E(l) \propto \sqrt{g/l} \implies E \propto l^{-1/2}$$

Amplitude follows $E \propto m\omega^2 A^2$, yielding $A \propto l^{-3/4}$.

5.3 Parametric Resonance (The Swing Effect)

If the parameter $l(t)$ is modulated periodically, specifically at $\Omega = 2\omega_0$, resonance occurs.

$$l(t) = l_0(1 + \epsilon \cos(2\omega_0 t))$$

This leads to the **Mathieu Equation**.

- **Mechanism:** If you shorten the pendulum when it is at the bottom (maximum tension/velocity) and lengthen it at the extremes (minimum tension), you do net positive work over a cycle.
- **Result:** The amplitude grows exponentially with time:

  $$\theta(t) \sim e^{st} \cos(\omega t)$$

  This is distinct from driven resonance (where $\Omega = \omega_0$) which grows linearly.