Physics Handout 6: Astronomy & Astrophysics Complete Formula Handbook

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"The cosmos is within us. We are made of star-stuff."
— Carl Sagan

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1 Celestial Coordinate Systems & Geometry

Definition: Coordinate Systems

Three fundamental coordinate systems are used in observational astronomy:

- **Horizon System**: Altitude a, Azimuth A (local, observer-dependent)
- **Equatorial System**: Right Ascension α, Declination δ (celestial, fixed)
- **Ecliptic System**: Longitude λ, Latitude β (orbital plane reference)

Observer latitude: ϕ

Theorem: Fundamental Altitude Formula

The altitude a of a celestial object at hour angle H and declination δ, observed from latitude ϕ, is given by:

$$\sin a = \sin\phi\sin\delta + \cos\phi\cos\delta\cos H$$

Parameters:

- a = altitude above horizon
- H = hour angle (LST - α)
- ϕ = observer's geographic latitude
- δ = object's declination

Key Formula: Rising and Setting Condition

An object rises or sets when $a = 0$. The hour angle at this moment satisfies:

$$\cos H_{\text{rise/set}} = -\tan\phi\tan\delta$$

Note: If $|\tan\phi\tan\delta| > 1$, the object is either circumpolar or never rises.

Lemma: Local Sidereal Time Relation

The Local Sidereal Time (LST) is related to Right Ascension and Hour Angle by:

$$\text{LST} = \alpha + H$$

This fundamental relation allows conversion between observer time and celestial coordinates.

Theorem: Spherical Law of Cosines

The angular separation θ between two points on the celestial sphere with coordinates (α_1, δ_1) and (α_2, δ_2) is:

$$\cos\theta = \sin\delta_1\sin\delta_2 + \cos\delta_1\cos\delta_2\cos(\alpha_1 - \alpha_2)$$

Olympiad Problem: Angular Separation

Two stars have equatorial coordinates:

$$(\alpha_1, \delta_1) = (6^{\text{h}}, 30^\circ), \quad (\alpha_2, \delta_2) = (7^{\text{h}}, -10^\circ)$$

Find their angular separation.

Solution Strategy:

1. Convert RA difference: $\Delta\alpha = 1^{\text{h}} = 15^\circ$
2. Apply spherical cosine law with $\delta_1 = 30^\circ$, $\delta_2 = -10^\circ$
3. Result: $\theta \approx 43.5^\circ$

2 Time Cycles & Orbital Periods

Key Formula: Synodic Period

The synodic period T_{syn} (observed from one body orbiting another) relates to individual sidereal periods:

$$\frac{1}{T_{\text{syn}}} = \left| \frac{1}{T_1} - \frac{1}{T_2} \right|$$

Application: Used for planetary conjunctions, lunar phases, and eclipse cycles.

Lemma: Metonic Cycle

For repeating astronomical cycles, find integers m, n such that:

$$mT_{\text{syn}} \approx nT_{\text{year}}$$

Method: Use continued fractions to find best rational approximation.

Olympiad Problem: Lunar Phase Repetition

The synodic month is $T_{\text{syn}} = 29.53$ days. Find the smallest integer number of years after which lunar phases repeat on the same calendar date.

Solution Strategy:

1. Compute ratio: $\dfrac{365.24}{29.53} \approx 12.368$
2. Use continued fraction expansion to find best approximation
3. The Metonic cycle gives: 19 years $\approx$ 235 lunar months

Answer: 19 years

3 Photometry & Magnitudes

Theorem: Pogson's Law

The relationship between magnitude difference and flux ratio is logarithmic:

$$m_1 - m_2 = -2.5 \log_{10}\left(\frac{F_1}{F_2}\right)$$

Consequence: A magnitude difference of 5 corresponds to a flux ratio of exactly 100.

Key Formula: Inverse Square Law

The observed flux F from a source of luminosity L at distance r is:

$$F = \frac{L}{4\pi r^2}$$

Important Note

Combining Pogson's Law with the inverse square law:

$$m - M = 5 \log_{10}\left(\frac{d}{10\,\text{pc}}\right)$$

where m is apparent magnitude, M is absolute magnitude, and d is distance in parsecs.

Olympiad Problem: Magnitude and Flux

A star appears 5 magnitudes brighter than another star. How many times larger is its observed flux?

Solution: Using Pogson's Law with $\Delta m = 5$:

$$5 = -2.5 \log_{10}\left(\frac{F_1}{F_2}\right) \implies \frac{F_1}{F_2} = 10^{-5/2.5} = 10^2 = 100$$

Answer: The flux is 100 times larger.

4 Stellar Physics

Definition: Initial Mass Function (IMF)

The IMF describes the distribution of stellar masses at formation:

$$\frac{dN}{dM} = kM^{-\alpha}$$

where $\alpha \approx 2.35$ (Salpeter value). **Interpretation:** Low-mass stars are exponentially more common than high-mass stars.

Key Formula: Logarithmic IMF

When considering equal logarithmic mass intervals:

$$\xi(\log M) = M\frac{dN}{dM} \propto M^{-(\alpha-1)} \propto M^{-1.35}$$

Theorem: Mass-Luminosity-Lifetime Relation

For main sequence stars, empirical relations give:

$$L \propto M^{\beta} \quad \text{where } \beta \approx 3.5 \text{ for solar-mass stars} \tag{1}$$

$$T_{\text{ms}} \propto \frac{M}{L} \propto M^{1-\beta} \propto M^{-2.5} \tag{2}$$

Consequence: Massive stars burn out much faster despite having more fuel.

Olympiad Problem: Main Sequence Lifetime Comparison

Compare the main sequence lifetimes of a $10M_\odot$ star and the Sun, given $L \propto M^{3.5}$.

Solution:

$$\frac{T_{10}}{T_\odot} = \left(\frac{M_{10}}{M_\odot}\right)^{1-3.5} = 10^{-2.5}$$
$$= \frac{1}{10^{2.5}} = \frac{1}{316} \approx 0.003$$

Answer: The $10M_\odot$ star lives only $\sim$30 million years if the Sun lives 10 billion years.

Important Note

For different mass ranges, the exponent β varies:

- $M < 0.43M_\odot$: $\beta \approx 2.3$
- $0.43M_\odot < M < 2M_\odot$: $\beta \approx 4.0$
- $2M_\odot < M < 55M_\odot$: $\beta \approx 3.5$
- $M > 55M_\odot$: $\beta \approx 1.0$

5 Cosmology

Definition: Redshift

The cosmological redshift z quantifies the stretching of wavelengths due to cosmic expansion:

$$z = \frac{\lambda_{\text{obs}} - \lambda_{\text{rest}}}{\lambda_{\text{rest}}} = \frac{\Delta\lambda}{\lambda_{\text{rest}}}$$

Key Formula: Scale Factor

The scale factor $a(t)$ relates to redshift by:

$$a = \frac{1}{1+z}$$

Normalization: $a_0 = 1$ at present time.

Theorem: Temperature Evolution

The CMB temperature scales inversely with the scale factor:

$$T(z) = T_0(1+z)$$

where $T_0 = 2.725\,\text{K}$ is the present-day CMB temperature.

Key Formula: Critical Density

The critical density ρ_c determines the geometry of space:

$$\rho_c = \frac{3H^2}{8\pi G}$$

Present value: $\rho_{c,0} \approx 9.47 \times 10^{-27}\,\text{kg/m}^3$

Theorem: Matter-Dominated Universe Scaling

In a matter-dominated universe ($\Omega_m = 1$):

$$\rho_m \propto a^{-3} \quad (3)$$

$$H \propto a^{-3/2} \propto t^{-1} \quad (4)$$

$$a \propto t^{2/3} \quad (5)$$

Olympiad Problem: CMB Temperature at High Redshift

A galaxy is observed at redshift $z = 6$. Calculate the CMB temperature at that epoch if $T_0 = 2.73\,\text{K}$.

Solution:

$$T = T_0(1+z) = 2.73 \times (1+6) = 2.73 \times 7 = 19.11\,\text{K}$$

Answer: $T \approx 19.1\,\text{K}$

Important Note

The Hubble parameter evolves with redshift as:

$$H(z) = H_0\sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda}$$

for a flat universe with matter and dark energy ($\Omega_k = 0$).

6 Astrometry: Parallax & Proper Motion

Key Formula: Parallax Distance

The distance to a star in parsecs is inversely proportional to its parallax angle:

$$d(\text{pc}) = \frac{1}{\pi(\text{arcsec})}$$

Definition: 1 parsec ≡ distance at which 1 AU subtends 1 arcsecond.

Definition: Proper Motion

Proper motion μ (in mas/yr or arcsec/yr) measures angular motion across the sky. Components:

$$\mu_\alpha \cos\delta = \text{motion in RA (corrected for declination)} \quad (6)$$

$$\mu_\delta = \text{motion in declination} \quad (7)$$

Total proper motion: $\mu = \sqrt{(\mu_\alpha \cos\delta)^2 + \mu_\delta^2}$

Lemma: Proper Motion from Velocity

For a star at position angle θ moving with transverse velocity $v_\perp$ at distance d:

$$\mu(\text{arcsec/yr}) = \frac{v_\perp(\text{km/s})}{4.74 \cdot d(\text{pc})}$$

Olympiad Problem: Proper Motion Components

A star at declination $\delta = 30°$ has proper motion $\mu = 200\,\text{mas/yr}$ at position angle 150° (measured east of north). Find μ_δ.

Solution: Position angle 150° means the motion vector makes 150° with north (toward declination).

$$\mu_\delta = \mu\cos(150°) = 200 \times (-0.866) = -173\,\text{mas/yr}$$

Answer: $\mu_\delta \approx -173\,\text{mas/yr}$ (southward motion)

Important Note

Tangential vs Radial Velocity:

- Tangential velocity: measured from proper motion
- Radial velocity: measured from Doppler shift
- Total space velocity: $v = \sqrt{v_r^2 + v_\perp^2}$

7 Planetary Geometry & Phases

Definition: Phase Angle

The phase angle ψ is the angle at the planet between the Sun and Earth.

Key Formula: Phase Geometry

For an inferior planet (Mercury, Venus) at distance r from Sun and Earth at distance r_1:

$$\cos\psi = \frac{r_1}{r}$$

where r is the instantaneous Earth-planet distance.

Lemma: Maximum Elongation

The maximum angular separation $\Delta\lambda_{\text{max}}$ of an inferior planet from the Sun satisfies:

$$\sin\Delta\lambda_{\text{max}} = \frac{a_{\text{planet}}}{a_\oplus}$$

where a denotes semi-major axis.

Olympiad Problem: Venus Half-Phase

Venus shows a half-phase (50% illuminated) as seen from Earth. Estimate its angular separation from the Sun.

Solution: Half-phase means $\psi = 90°$ (Sun-Venus-Earth forms right angle).
For Venus: $a_V \approx 0.72\,\text{AU}$, $a_\oplus = 1\,\text{AU}$
When $\psi = 90°$, the configuration is near maximum elongation:

$$\sin\Delta\lambda \approx \frac{0.72}{1} = 0.72 \implies \Delta\lambda \approx 46°$$

Answer: Venus appears approximately 46° from the Sun at half-phase.

Important Note

Phases of Planets:

- **Inferior planets** (Mercury, Venus): Show full range of phases like the Moon
- **Superior planets** (Mars, Jupiter, etc.): Only show gibbousnearly-full phases
- Phase fraction illuminated: $\Phi = \frac{1+\cos\psi}{2}$

Olympiad Strategy & Final Tips

Problem-Solving Framework

Step 1: Identify the physical regime

- Coordinate transformation?
- Orbital mechanics?
- Photometric calculation?

Step 2: Write down all given quantities with correct units
Step 3: Apply relevant theorem/formula from colored boxes
Step 4: Perform dimensional analysis as a check
Step 5: Verify answer makes physical sense (order of magnitude, limiting cases)

Common Olympiad Pitfalls

1. **Unit confusion:** Always check degrees vs radians, AU vs parsecs
2. **Reference frames:** Be explicit about observer location
3. **Sign conventions:** Particularly in coordinate transformations
4. **Logarithms:** Remember magnitude scale is backwards (bright = small)
5. **Small angle approximation:** Valid only for $\theta < 0.1$ rad $\approx 6°$

"Somewhere, something incredible is waiting to be known."
— Carl Sagan

Good luck with your olympiad preparation!