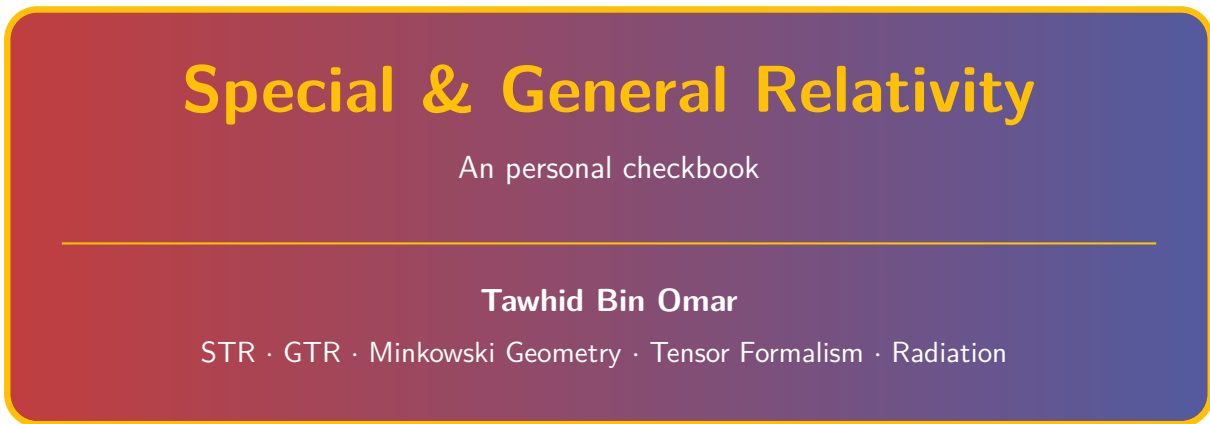


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**Topics Covered:** Lorentz Transformations · Minkowski Geometry · Relativistic Dynamics · Four-Vectors · Tensor Formalism · Electrodynamics in STR · Multipole Expansion · Radiation Theory (Larmor) · General Relativity · Schwarzschild Metric · Geodesic Equations

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# 1 Special Theory of Relativity (STR)

## Context: Motivation

Classical mechanics postulates that all mechanical processes are invariant under the Galilean group (transformations between inertial reference frames). However, Maxwell's equations are **not** Galilean-invariant — they are invariant under the **Lorentz group**. To resolve this contradiction, the laws of mechanics must be modified to become Lorentz-invariant.

## 1.1 Lorentz Transformations

### Key Formula: Lorentz Transformations (boost along $x$ -axis)

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The notation  $\beta = v/c$ ,  $\gamma = (1 - \beta^2)^{-1/2}$  is standard.

### Remark: Symmetry of Space and Time

The Lorentz transformation treats  $x$  and  $ct$  symmetrically. This “mixing” of space and time requires abandoning Euclidean geometry in favour of **Minkowski geometry**.

## 1.2 Relativistic Velocity Addition

### Key Formula: Relativistic Velocity Addition

Parallel component:

$$U = \frac{U'_{\parallel} + v}{1 + \frac{U'_{\parallel} v}{c^2}}$$

Transverse component:

$$\frac{dy}{dt} = \frac{v'_y \sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{v v'_x}{c^2}}$$

### Remark: Relativity of Simultaneity

Events simultaneous in frame  $K$  are **not** simultaneous in  $K'$ . This, together with velocity addition, forces us to use Minkowski geometry.

## 1.3 Minkowski Spacetime and the Interval

### Definition: Event

An **event** is a particle's spatial and temporal position in a chosen reference frame — specified by four degrees of freedom:  $(ct, x_1, x_2, x_3)$ .

**Definition: Spacetime Interval**

The invariant interval in Minkowski spacetime:

$$ds^2 = c^2 dt^2 - dr^2 = c^2 dt^2 - dx_1^2 - dx_2^2 - dx_3^2$$

**Theorem: Causal Structure of Spacetime**

- $ds^2 > 0$  (**timelike**): One can find a frame where events occur at the same spatial point.
- $ds^2 < 0$  (**spacelike**): One can find a frame where events are simultaneous. No causal relationship is possible.
- $ds^2 = 0$  (**lightlike/null**): The events are connected by a light signal.

**Key Formula: Proper Time**

For a particle moving with velocity  $v$ , the proper time  $d\tau$  measured in its rest frame relates to coordinate time by:

$$d\tau = dt \sqrt{1 - \frac{v^2}{c^2}}$$

**1.4 Exercises: STR Kinematics****Exercise 1 (MSU) — Length Contraction and Time Dilation**

A rod flies past a stationary mark at constant speed. The flight time in frame  $S$  is  $\Delta t = 20$  ns. In the frame  $S'$  (fixed to the rod), the mark moves along it for  $\Delta t' = 25$  ns. Find the **proper length** of the rod.

**Exercise 2 (MSU) — Unstable Particle**

An unstable particle has proper lifetime  $\Delta t_0 = 10$  ns. In the lab frame its lifetime is  $\Delta t = 20$  ns. Find the **distance** travelled before decaying.

**Exercise 3 (MSU) — Twin Paradox / Clock Drift**

An interstellar ship travels to a star  $L = 3$  light-years away at  $V = 5 \times 10^6$  m/s, then returns at the same speed. By what time interval  $\Delta t$  will the ship's clock **lag behind** Earth clocks upon return? (Ignore acceleration/deceleration.)

**Exercise 4 (MSU) — Lorentz Contraction of an Angled Rod**

A rod moves at  $V = c/2$  relative to frame  $S$  with proper length  $\ell = 1$  m. In the rest frame  $S'$  of the rod, the angle between the rod and its direction of motion is  $\phi = 45^\circ$ . Find the length  $\ell$  and angle  $\phi$  in frame  $S$ .

**Exercise 5 (MSU) — Spacetime Interval Analysis**

Two events:  $A : (x_1, t_1) = (0 \text{ m}, 0 \text{ s})$ ;  $B : (x_2, t_2) = (5 \text{ m}, 10^{-8} \text{ s})$ .

- (1) Find the frame where the spatial separation is **minimum**. Find that distance and the frame's speed.
- (2) Find the frame where the time interval is **minimum**. Find that interval and the frame's speed.
- (3) Can these events be in a **cause-and-effect** relationship?

**Exercise 6 (MSU) — Searchlight Aberration**

Two searchlights emit beams in opposite directions along the  $Y$ -axis. At what speed  $U$  (perpendicular to  $Y$ ) must they move so that the beams make an angle  $\alpha = 90^\circ$  to each other?

**Problem 1 (Parfenov — Astronomical Aberration)**

Light from a “fixed” star falls at angle  $\alpha$  to the plane of Earth's orbit (in the Sun's rest frame). The Earth moves at  $v \approx 30 \text{ km/s}$ . Find the change in the telescope's pointing angle between positions where Earth moves **towards** and **away from** the incoming light.

**Problem 2 (Lightman — Tachyons)**

A tachyon source emits particles at speed  $u > c$  (in its rest frame). A signal is sent to an observer at rest at distance  $L$ .

- (a) After what time is a tachyon reply received?
- (b) What if the observer moves away at speed  $v$  and sends a reply from distance  $L$ ?

**Problem 3 (Lightman — Relativistic Aberration of Stars)**

In the rest frame of the fixed stars, their distribution is isotropic:  $dN = N (d\Omega/4\pi)$ . An observer moves at relativistic speed  $\beta$  in the  $e_x$  direction. Find the distribution density  $P(\theta', \phi')$  in the moving frame. Verify  $\int P d\Omega' = N$  and  $P \rightarrow N/4\pi$  as  $\beta \rightarrow 0$ . *Where does a clustering of stars appear?*

**Problem 4 (Physics Cup — Accelerating Spaceship)**

A spaceship moves with constant proper acceleration  $g$  and launches two rockets in its direction of motion at speeds  $v$  and  $2v$  (in the ship's frame), both comparable to  $c$ . Find the time interval in the ship's frame between its encounters with the two rockets.

## 2 Relativistic Dynamics

### Key Formula: Relativistic Momentum and Energy

Redefining momentum to ensure Lorentz invariance:

$$\vec{p} = \frac{m_0 \vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Total energy:

$$E = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Kinetic energy:

$$E_k = m_0 c^2 \left( \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right)$$

### Key Formula: Energy-Momentum Relation (Invariant)

$$E^2 - c^2 p^2 = m_0^2 c^4$$

Also useful:  $\vec{p} = \frac{E\vec{v}}{c^2}$

### Remark: Massless Particles

For a particle with  $m_0 = 0$  (e.g. the photon): non-zero energy is possible only if it moves **exactly at the speed of light**.

### 2.1 Relativistic Action and Lagrangian

#### Definition: Relativistic Action

By analogy with the geometric problem of minimal length:

$$S = -\alpha \int_A^B ds = -mc \int_A^B ds$$

This yields the **relativistic Lagrangian**:

$$L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}}$$

From which canonical momentum and energy follow by standard methods.

## 2.2 Exercises and Problems: Relativistic Dynamics

**Exercise 1 — Electron in a Transverse Electric Field**

An electron enters with speed  $v_0$  into a uniform electric field  $\mathcal{E}$  perpendicular to its velocity. Find the law governing the change in the **magnitude** of the electron's velocity. (Neglect radiation.)

**Exercise 2 — Photon Absorption/Emission by an Electron**

Prove that an **isolated** electron can neither absorb nor emit a photon.

**Exercise 3 — Maximum Scattering Angle**

Determine the **maximum scattering angle** of a particle of mass  $m_1$  incident on a stationary particle of mass  $m_2$ .

**Problem 1 (Parfenov — Neutral Particle Decay / Pair Creation)**

Two electron-positron pairs are created by two photons. The total energies of each pair are  $E_1$  and  $E_2$ . The photon lines intersect at a point with angle  $\theta$  between them. Hypothesize these photons came from a neutral particle decaying at that point. **Find the mass of this particle.**

**Problem 2 (Parfenov — Higgs Boson Threshold)**

Antiprotons of energy  $E$  strike a stationary proton target; Higgs bosons ( $M$ ) may be produced. Proton rest mass:  $m$ . Find the **minimum energy**  $E$  for the reaction to occur.

**Problem 3 (Parfenov — Completely Inelastic Relativistic Collision)**

A particle of mass  $m$  and Lorentz factor  $\gamma$  collides inelastically with an identical particle at rest. Find the **mass**  $m'$  and **velocity**  $v'$  of the resulting particle.

**Problem 4 (Parfenov — Three-Body Decay)**

A stationary particle of mass  $M = nm$  decays into three particles of masses  $m, 2m, 3m$ . Find the **maximum possible energy** of one of the particles.

**Problem 5 (Physics Cup — Electron-Positron Annihilation Angle)**

An electron (initially at rest) is accelerated through voltage  $U = \frac{km_0c^2}{e}$ . It collides with a stationary positron and annihilates into two photons. Find the **minimum angle**  $\alpha$  between the photon emission directions. Express in terms of  $k$ ; evaluate for  $k = 1$ .

### 3 Particle Motion in Fields

#### 3.1 Motion in a Uniform Electric Field

##### Key Formula: Relativistic Motion in Electric Field

For charge  $e$  in uniform field  $\mathcal{E}$  (horizontal):

$$p_x = e\mathcal{E}t, \quad p_y = p_0$$

Trajectory (catenary):

$$x = \frac{E_0}{e\mathcal{E}} \cosh\left(\frac{e\mathcal{E}y}{p_0c}\right)$$

#### 3.2 Motion in a Uniform Magnetic Field

##### Key Formula: Relativistic Cyclotron Frequency and Radius

Equation of motion:  $\frac{d\vec{p}}{dt} = \frac{e}{c}[\vec{v} \times \vec{H}]$

Introducing  $\omega = \frac{eH}{E}$ , the solution is circular motion:

$$x = x_0 + r \sin(\omega t + \alpha), \quad y = y_0 + r \cos(\omega t + \alpha)$$

Cyclotron radius:  $r = \frac{cp_{\perp}}{eH}$

##### Key Formula: Zeeman-Split Oscillation Frequencies

For a charged oscillator with natural frequency  $\omega_0$  in magnetic field  $H$ :

$$\omega = \sqrt{\omega_0^2 + \frac{1}{4}\left(\frac{eH}{mc}\right)^2} \pm \frac{eH}{2mc}$$

##### Exercise 1 (Landau & Lifshitz — Parallel $\vec{E}$ and $\vec{B}$ )

Determine the relativistic motion of a charged particle in **parallel** uniform electric and magnetic fields.

##### Exercise 2 (Landau & Lifshitz — Crossed Fields)

Determine the relativistic motion of a charge in mutually **perpendicular** electric and magnetic fields of equal magnitude.

##### Exercise 3 (200 Puzzling Physics Problems)

An electron moving at  $0.6c$  enters a uniform electric field perpendicular to its velocity. On exit, the velocity makes a  $45^\circ$  angle with the original direction.

- (a) Find the exit speed  $v_1$ .
- (b) Find the displacement  $d$  if  $\mathcal{E} = 510$  kV/m. (Rest energy:  $m_e c^2 = 510$  keV.)

**Exercise 4 (Parfenov / Lightman — Doppler Effect on a Moving Mirror)**

Light of frequency  $\omega_0$  falls at angle of incidence  $\alpha$  on a mirror moving perpendicularly to its surface at speed  $v$ . Find the **frequency of reflected light** and the **angle of reflection**.

## 4 Four-Dimensional Tensor Formalism

### Definition: Four-Velocity

$$u^0 = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad u^1 = \frac{v_x}{c} \cdot \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad u^2 = \frac{v_y}{c} \cdot \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad u^3 = \frac{v_z}{c} \cdot \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

### Definition: Minkowski Metric Tensor

The interval  $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$  with:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

(Flat Minkowski space.) Repeated indices (one up, one down) imply summation: **Einstein convention**.

### Definition: Electromagnetic Field Tensor

Using the four-potential  $(\phi, A^1, A^2, A^3)$ :

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

### Definition: Four-Momentum

$$p^\mu = \left( \frac{E}{c}, p_x, p_y, p_z \right) = m_0 c u^\mu$$

### Key Formula: Lorentz Transformation (Boost along $x$ )

Matrix form:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

**Tensor transformation:**  $F'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F^{\alpha\beta}$

**Key Formula: Lorentz Transformation of Fields**

$$\vec{E}' = \gamma(\vec{E} + [\vec{v} \times \vec{B}]) + (1 - \gamma) \frac{\vec{v}(\vec{v} \cdot \vec{E})}{v^2}$$

$$\vec{B}' = \gamma\left(\vec{B} - \frac{[\vec{v} \times \vec{E}]}{c^2}\right) + (1 - \gamma) \frac{\vec{v}(\vec{v} \cdot \vec{B})}{v^2}$$

**Key Formula: Four-Dimensional Newton's Equation (in EM field)**

$$\frac{dp^\mu}{ds} = \frac{q}{c} F^{\mu\nu} u_\nu$$

Equivalent in 3D to:

$$\frac{d\vec{p}}{dt} = q(\vec{E} + [\vec{v} \times \vec{B}]), \quad \frac{dE}{dt} = q(\vec{E} \cdot \vec{v})$$

**Key Formula: Maxwell's Equations (Covariant Form)**

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \frac{4\pi}{c} j^\mu$$

where the four-current is  $j^\mu = (c\rho, j_x, j_y, j_z)$ .

**4.1 Problems: Four-Dimensional Formalism****Problem 1 (Parfenov — Charge near Fixed Charge)**

A particle with charge  $q = -e$  moves near a fixed charge  $+Q$ . Neglecting radiation, investigate the possible trajectories.

**Problem 2 (Parfenov — Cylinder in Particle Cloud)**

A cylinder of radius  $R$  moves at speed  $v$  along its symmetry axis through a cloud of particles (mass  $m$ , concentration  $n$ ; particle speeds  $\ll v$  in frame  $K$ ). Collisions are elastic. Find the **deceleration force** in frame  $K$ .

**Problem 3 (Parfenov — Sphere in Particle Cloud)**

A sphere of radius  $R$  moves at speed  $v$  through the same particle cloud.

- Find the deceleration force (elastic collisions).
- Find the angular distribution of reflected particles.

**Problem 4 (Parfenov — Doppler in a Medium)**

A beam of excited atoms ( $E = \hbar\omega_0$  photon transition) moves at speed  $v$  in a medium with refractive index  $n = n(\omega)$ . An atom emits a photon at angle  $\theta$  to its direction of motion. Find the **observed frequency** of the photon. ( $E_\gamma \ll m_0c^2$ .)

**Problem 5 (Parfenov — Charged Rod and Moving Charge)**

An infinitely long rod with linear charge density  $\lambda$  (in its rest frame) moves along itself at speed  $v$ . A point charge  $q$  moves at the same speed  $v$  at distance  $r$  from the rod. Find the **force on the charge**.

## 5 Field Theory: Moving Charges and Radiation

### 5.1 Multipole Expansion

#### Key Formula: Multipole Expansion of Electric Potential

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left[ \frac{Q}{r} + \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} + \frac{1}{2} \sum_{i,j} Q_{ij} \frac{x_i x_j}{r^5} + \dots \right]$$

Dipole moment:  $\mathbf{p} = \int \mathbf{r}' \rho(\mathbf{r}') d^3 r'$

Quadrupole tensor:  $Q_{ij} = \int (3x'_i x'_j - r'^2 \delta_{ij}) \rho(\mathbf{r}') d^3 r'$

### 5.2 Radiation by Accelerating Charges

#### Key Formula: Electric Field in the Radiation Zone

$$\mathbf{E} = \frac{1}{c^2 r} [\mathbf{n} \times [\mathbf{n} \times \ddot{\mathbf{p}}(t_r)]], \quad |\mathbf{E}| = \frac{\ddot{p} \sin \theta}{c^2 r}$$

where  $\theta$  is the angle between  $\ddot{\mathbf{p}}$  and  $\mathbf{n}$ , and  $t_r = t - r/c$  is the retarded time.

Magnetic field:  $\mathbf{B} = [\mathbf{n} \times \mathbf{E}]$

#### Remark: Condition for Radiation

Since  $\mathbf{p} = \sum_i q_i \mathbf{r}_i$ , wave radiation occurs **only when a charged particle accelerates**.

#### Key Formula: Larmor Formula (Radiated Power)

CGS:  $I = \frac{2\ddot{p}^2}{3c^3}$

SI:  $I = \frac{\mu_0 \ddot{p}^2}{6\pi c} = \frac{\ddot{p}^2}{6\pi\epsilon_0 c^3}$

## 6 General Theory of Relativity (GTR)

### Context: From STR to GTR

STR describes motion in **flat** Minkowski space within inertial frames. GTR extends this by asking: how do we describe motion in non-inertial frames and in the presence of gravity?

Key insight: inertial forces (e.g. in a rotating frame) and gravity are both potential-type. GTR reframes gravity not as a force but as the **curvature of spacetime**, caused by the presence of mass-energy.

### Definition: Rotating Frame Metric

Example of a non-flat metric (rotating frame, cylindrical coordinates):

$$ds^2 = - \left[ 1 - \frac{\omega^2 r^2}{c^2} \right] c^2 dt^2 + 2\omega r^2 d\varphi dt + dr^2 + r^2 d\varphi^2 + dz^2$$

### 6.1 Geodesic Equation

#### Key Formula: Geodesic Equation (Equation of Motion in Curved Spacetime)

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\nu\theta}^\mu \frac{dx^\nu}{d\tau} \frac{dx^\theta}{d\tau} = 0$$

Christoffel symbols:

$$\Gamma_{\nu\theta}^\mu = \frac{1}{2} g^{\mu\rho} \left( \frac{\partial g_{\nu\rho}}{\partial x^\theta} + \frac{\partial g_{\theta\rho}}{\partial x^\nu} - \frac{\partial g_{\nu\theta}}{\partial x^\rho} \right)$$

#### Remark: Recovering Classical Mechanics

In the classical limit ( $c \rightarrow \infty$ , low velocity), the geodesic equation in the rotating frame metric reproduces Newton's equations of motion including **Coriolis** and **centrifugal** terms — confirming the geometric description of inertial forces.

#### Remark: Gravitational Lensing

Light, as an electromagnetic wave, follows the geodesic equation. Near a massive object, the curved spacetime deflects the path of light — this is **gravitational lensing**.

### 6.2 Schwarzschild Metric

#### Definition: Schwarzschild Metric (Spherical Symmetry)

The simplest non-flat solution — spacetime curved by a spherically symmetric mass  $M$ :

$$ds^2 = - \left( 1 - \frac{2GM}{c^2 r} \right) c^2 dt^2 + \left( 1 - \frac{2GM}{c^2 r} \right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2$$

**Key Formula: Non-Zero Christoffel Symbols (Schwarzschild)**

$$\begin{aligned} \Gamma_{tr}^t = \Gamma_{rt}^t &= \frac{GM}{c^2 r^2} \left(1 - \frac{2GM}{c^2 r}\right)^{-1} & \Gamma_{tt}^r &= \frac{GM}{c^2 r^2} \left(1 - \frac{2GM}{c^2 r}\right) \\ \Gamma_{rr}^r &= -\frac{GM}{c^2 r^2} \left(1 - \frac{2GM}{c^2 r}\right)^{-1} & \Gamma_{\theta\theta}^r &= -r \left(1 - \frac{2GM}{c^2 r}\right) \\ \Gamma_{\varphi\varphi}^r &= -r \left(1 - \frac{2GM}{c^2 r}\right) \sin^2 \theta & \Gamma_{r\theta}^\theta = \Gamma_{\theta r}^\theta &= \frac{1}{r} \\ \Gamma_{\varphi\varphi}^\theta &= -\sin \theta \cos \theta & \Gamma_{r\varphi}^\varphi = \Gamma_{\varphi r}^\varphi &= \frac{1}{r} \\ \Gamma_{\theta\varphi}^\varphi = \Gamma_{\varphi\theta}^\varphi &= \cot \theta \end{aligned}$$

**Key Formula: First Integrals (Equatorial Plane  $\theta = \pi/2$ )****Energy conservation:**

$$E = \left(1 - \frac{2GM}{c^2 r}\right) c^2 \frac{dt}{d\tau} = \text{const}$$

**Angular momentum conservation:**

$$L = r^2 \frac{d\varphi}{d\tau} = \text{const}$$

**Radial equation:**

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{c^2} - \left(1 - \frac{2GM}{c^2 r}\right) \left(c^2 + \frac{L^2}{r^2}\right)$$

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Special &amp; General Relativity Olympiad Handout